XLI. *A Method of finding, by the Help of Sir* Isaac Newton's *binomial Theorem, a near Value of the very slowly-converging infinite Series* $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.$ *when* x *is very nearly equal to* 1. *By* Francis Maseres, *Esq. F. R. S. Cursitor Baron of the Exchequer.*

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ARTICLE I.

IF the capital letters A, B, C, D, E, &c. be put for the numeral co-efficients of the powers of x in the said series, so that A shall be $= 1$, $B = \frac{1}{2}$, $C = \frac{1}{3}$, $D = \frac{1}{4}$, $E = \frac{1}{5}$, and so on, we shall have $B = \frac{1}{2} \times A$, $C = \frac{2}{3} \times B$, $D = \frac{3}{4} \times C$, $E = \frac{4}{5} \times D$, &c. and the series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c$ will be $= x + \frac{1}{2}Axx + \frac{2}{3}Bx^3 + \frac{3}{4}Cx^4 + \frac{4}{5}Dx^5 + \frac{5}{6}Ex^6 + \frac{6}{7}Fx^7 + \frac{7}{8}Gx^8 + \&c.$; in which series the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, &c. which generate the co-efficients of the powers of x in the several terms after the first term x, are derived from each other by the continual addition of 1 to both their numerators and their denominators.

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2. This obſervation ſuggeſts to us a method of finding a near value of the ſum of this ſeries by the help of Sir ISAAC NEWTON's binomial theorem, which may be explained as follows.

If m and n repreſent any two whole numbers, the reciprocal of the $\frac{m}{n}$th power of the binomial quantity $1-x$, or, according to Sir ISAAC NEWTON's notation of powers, the quantity $\overline{1-x}|^{\frac{-m}{n}}$, will , according to that celebrated theorem, be equal to the infinite ſeries

$$1+\frac{m}{n}\times x$$
$$+\frac{m}{n}\times\frac{m+n}{2n}\times xx+\frac{m}{n}\times\frac{m+n}{2n}\times\frac{m+2n}{3n}\times x^3$$
$$+\frac{m}{n}\times\frac{m+n}{2n}\times\frac{m+2n}{3n}\times\frac{m+3n}{4n}\times x^4$$
$$+\frac{m}{n}+\frac{m+n}{2n}\times\frac{m+2n}{3n}\times\frac{m+3n}{4n}\times\frac{m+4n}{5n}\times x^5+\text{\&c.}$$

or $1+\frac{m}{n}\text{A}x+\frac{m+n}{2n}\text{B}xx+\frac{m+2n}{3n}\text{C}x^3+\frac{m+3n}{4n}\text{D}x^4+\frac{m+4n}{5n}\text{E}x^5$ $+\frac{m+5n}{6n}\text{F}x^6+\frac{m+6n}{7n}\text{G}x^7+\frac{m+7n}{8n}\text{H}x^8+\text{\&c.}$; in which ſeries the capital letters A, B, C, D, E, F, G, H, &c. ſtand for 1 and the co-efficients of x, xx, x^3, x^4, x^5, x^6, x^7, x^8, &c. Now it is evident, that the generating fractions $\frac{m+n}{2n}$, $\frac{m+2n}{3n}$, $\frac{m+3n}{4n}$, $\frac{m+4n}{5n}$, $\frac{m+5n}{6n}$, $\frac{m+6n}{7n}$, $\frac{m+7n}{8n}$, &c. are derived from $\frac{m}{n}$ and from each other by the continual addition of n to both their numerators and denominators. Therefore, though they are greater than they would be if m was ſubtracted from the numerator

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of each of them, that is, than the fractions $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, $\frac{4n}{5n}$, $\frac{5n}{6n}$, $\frac{6n}{7n}$, $\frac{7n}{8n}$, &c. and consequently, than the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, &c. which are respectively equal to $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, $\frac{4n}{5n}$, $\frac{5n}{6n}$, $\frac{6n}{7n}$, $\frac{7n}{8n}$, &c. Yet, the further we go in the series, the less is the proportion in which they exceed the latter fractions; insomuch that, if we go far enough in the series, we may find terms in it whose proportion to the corresponding terms in the series $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, &c. shall approach as near as we please to a proportion of equality. And, by taking n of a very great magnitude in comparison of m, we may even make the first terms of the series $\frac{m+n}{2n}$, $\frac{m+2n}{3n}$, $\frac{m+3n}{4n}$, $\frac{m+4n}{5n}$, $\frac{m+5n}{6n}$, $\frac{m+6n}{7n}$, $\frac{m+7n}{8n}$, &c. approach very nearly to an equality with the corresponding terms of the series $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, &c. which are the generating fractions of the proposed series $x+\frac{xx}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}+\frac{x^8}{8}+$ &c. In order to this, let m be taken $=1$, and $n=1,000,000,000,000$, that is, $=$ a billion, or the square of a million, which, to avoid the frequent repetition of so many cyphers, we will call b.

Then will $\overline{1-x}|^{-\frac{1}{1,000,000,000,000}}$, or $\overline{1-x}|^{-\frac{1}{b}}$, or $\frac{1}{\overline{1-x}|^{\frac{1}{b}}}$, be

$$=1+\frac{1}{b}Ax+\frac{1+b}{2b}Bxx+\frac{1+2b}{3b}Cx^3+\frac{1+3b}{4b}Dx^4+\frac{1+4b}{5b}Ex^5$$

$+\frac{1+5b}{6b}Fx^6+\frac{1+6b}{7b}Gx^7+\frac{1+7b}{8b}Hx^8+$ &c. which, on account of the

the great magnitude of b, $2b$, $3b$, $4b$, $5b$, $6b$, $7b$, &c. in comparison of 1, will be almost equal to (though somewhat greater than) $1 + \frac{1}{b}Ax + \frac{b}{2b}Bxx + \frac{2b}{3b}Cx^3 + \frac{3b}{4b}Dx^4 + \frac{4b}{5b}Ex^5 + \frac{5b}{6b}Fx^6 + \frac{6b}{7b}Gx^7 + \frac{7b}{8b}Hx^8 +$ &c. or $1 + \frac{1}{b}Ax + \frac{1}{2}Bxx + \frac{2}{3}Cx^3 + \frac{3}{4}Dx^4 + \frac{4}{5}Ex^5 + \frac{5}{6}Fx^6 + \frac{6}{7}Gx^7 + \frac{7}{8}Hx^8 +$ &c. or $1 + \frac{1}{b} \times 1 \times x + \frac{1}{2} \times \frac{1}{b} \times 1 \times xx + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{b} \times 1 \times x^3 + \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{b} \times 1 \times x^4 +$ &c. or $1 + \frac{1}{b}x + \frac{1}{b} \times \frac{1}{2}xx + \frac{1}{b} \times \frac{1}{3} \times x^3 + \frac{1}{b} \times \frac{1}{4}x^4 +$ &c. or $1 + \frac{x}{b} + \frac{xx}{2b} + \frac{x^3}{3b} + \frac{x^4}{4b} + \frac{x^5}{5b} + \frac{x^6}{6b} + \frac{x^7}{7b} + \frac{x^8}{8b} +$ &c. Therefore, multiplying both sides by b, we shall have $b \times \frac{1}{\overline{1-x}|^{\frac{1}{b}}}$ nearly $= b + x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} +$ &c.; and, subtracting b from both sides, $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} +$ &c. nearly $= b \times \frac{1}{\overline{1-x}|^{\frac{1}{b}}} - b = b \times \overline{\frac{1}{1-x}}|^{\frac{1}{b}} - b$; that is, the proposed series $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} +$ &c. will be nearly equal to $b \times \overline{\frac{1}{1-x}}|^{\frac{1}{b}} - b$. We must therefore first subtract x from 1, and then divide 1 by the remainder, which will give us a quotient equal to $\frac{1}{1-x}$. And, having found this quotient, we must extract its bth, or 1,000,000,000,000th, root, and multiply the said root by b, or 1,000,000,000,000; and, lastly, from the product we must subtract b, or 1,000,000,000,000: and the

the remainder thereby obtained will be nearly equal to the propoſed infinite ſeries $x+\frac{xx}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}+$ &c. Q. E. I.

An example of the foregoing method of ſumming the ſaid infinite ſeries.

3. As an example of this method of finding the value of the ſeries $x+\frac{xx}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+$ &c. let us ſuppoſe x to be equal to $\frac{9}{10}$.

Then we ſhall have $1-x=1-\frac{9}{10}=\frac{1}{10}$, and $\frac{1}{1-x}=10$. Now, ſince the logarithm of 10 in BRIGGS's Syſtem of logarithms is 1, the logarithm of the 1,000,000,000,000th root of 10 muſt be the 1,000,000,000,000th part of 1, or muſt be = .000,000,000,001. This logarithm is too ſmall to be found in the common tables of logarithms, which go only to ſeven places of figures; and therefore the number correſponding to it, that is, the 1,000,000,000,000th root of 10, cannot be found by the help of thoſe tables; but it may be found in the manner following. The 1,000,000,000,000th root of 10 is a number that is ſomewhat, and but a very little, greater than 1. That number, therefore, and 1 will repreſent two ordinates to the axis, or aſymptote, of a logarithmick curve that are very nearly contiguous to each other: whence it follows, that the ſub-tangent of the curve will

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bear very nearly the ſame proportion to the leſſer ordinate 1, as the abſciſs of the axis intercepted between the two ordinates, that is, as the logarithm of the ratio of the greater ordinate to the leſſer, or the logarithm .000,000,000,001, bears to the difference of the ſaid ordinates. Say therefore, as .434,294,481,9 (which is the ſub-tangent of the logarithmick curve in BRIGGS's Syſtem of Logarithms) is to 1 (or the leſſer of the two ordinates) ſo is .000,000,000,001 to a fourth number, which will be .000,000,000,002,302,585,093; and this fourth number will be the exceſs of the greater of the ſaid two ordinates above the leſſer, or of the billionth root of 10 above 1. Therefore the billionth root of 10 will be = 1.000,000,000,002,302,585,093; which, being multiplied by 1,000,000,000,000, will be = 1,000,000,000,002.302,585,093; from which if we ſubtract 1,000,000,000,000, the remainder will be 2.302,585,093. Therefore 2.302,585,093 is nearly equal to the infinite ſeries $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7}$ + &c. when x is $= \frac{9}{10}$. Q. E. I.

4. This number 2.302,585,093 gives the value of the ſeries $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} +$ &c. exact to nine places of figures, the error being only in the 10th figure 3, which ought to be a 2 inſtead of a 3, the more accurate value of that ſeries (which is equal to the logarithm of

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the ratio of 1 to $1 - x$ in NAPIER's Syſtem of Logarithms, that is, in the preſent example, to the logarithm of the ratio of 1 to $\frac{1}{10}$, or of 10 to 1, or to NAPIER's logarithm of 10) being 2.302,585,092,994,04.

5. I believe that ſimilar applications of the binomial theorem may be made for the ſummation of other ſlowly-converging infinite ſerieſes, whenever the generating fractions (by the multiplication of which the numeral co-efficients of the terms of ſuch ſerieſes are produced from each other) are formed by the addition of a given number to both their numerators and denominators. In our endeavours, therefore, to ſum ſuch ſerieſes it will be proper to attend to the law of the ſaid generating fractions.